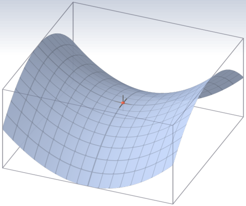

Morse Theory and the Cosmic Skeleton

Presentation for the “Cosmic Web“ course

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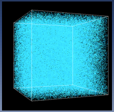


Overview

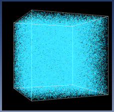
Landscape analogy

Figure: Density field as shadowed landscape [ACPvdWS10]

Presented approach treats density field as an manifold, created by making scalar field (e.g. density) value a spatial coordinate (hight in 3D case).

Analysis focuses on **critical points** which are either

- peaks – local maxima
- basins – local minima
- passes – saddle points



Some Definitions

n-disk
$$D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}, \; \mathrm{int}(D^n) = \{x \in \mathbb{R}^n : |x| < 1\}$$

n-cell
space homeomorphic to the open n-disk $\mathrm{int}(D^n)$

differentiable manifold
locally similar enough to a linear space to allow one to do calculus

homeomorphism
exists bijection $f : X \to Y$, f and f^{-1} continuous

diffeomorfism
and f is differentiable

[Kuk11]

Some Definitions

homotopy equivalent continuous transition

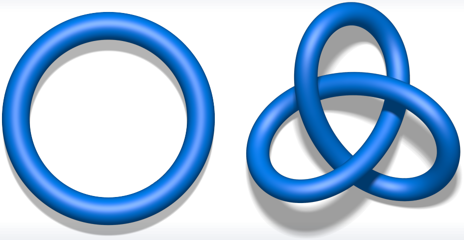
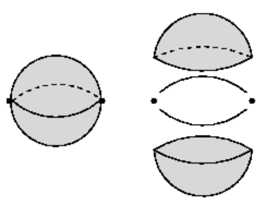


Figure: Homeomorphic but not homotopy equivalent

CW-complex

CW-complex is a pair (X, ε) of Hausdorff space (manifold) X and its cell decomposition $\varepsilon = \{e_\alpha | \alpha \in I\}$, where e_α is an **open** cell of some dimension, and

$$X = \coprod_{\alpha \in I} e_\alpha$$



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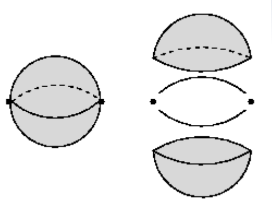

which satisfy

- For each n-cell $e \in \varepsilon$ there is a map $\Phi_e : D^n \to X$ taking $\mathrm{int}(D^n) \to e$ and S^{n-1} into X^{n-1}
- For any cell $e \in \varepsilon$ the closure $\bar{\varepsilon}$ intersects only a finite number of cells in ε.
- $A \subseteq X$ is closed iff $A \cap \bar{e}$ is closed in X for each $e \in \varepsilon$.

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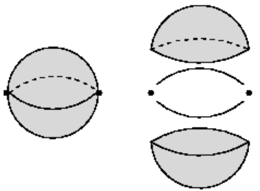

n-skeleton of X

$$X^n = \coprod_{\alpha \in I : \dim(e_\alpha) \leq n} e_\alpha$$

Critical Points

- M^n – differentable manifold
- $f : M^n \to \mathbb{R}$ – smooth function

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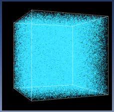
Critical point x is **not degenerated** if its Hessian is not singular.
Function f is called **Morse function** if all critical points are not degenerated.



Critical Points

Index of a critical point

The Index of a critical point is a dimension of the negative-definite submatrix of the hessian matrix **H** calculated at that point.

$$\mathbf{H}_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

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basins index $= 0$

passes index $= 1$

peaks index $= 2$

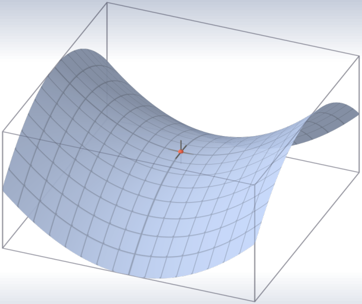
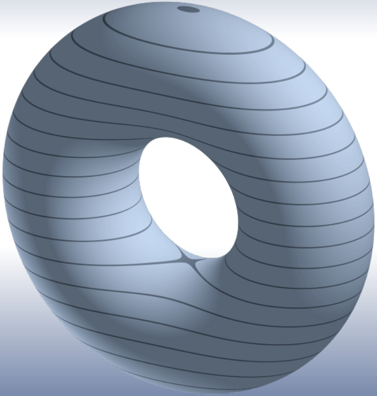


Figure: Saddle point, index $= 1$

Theorems

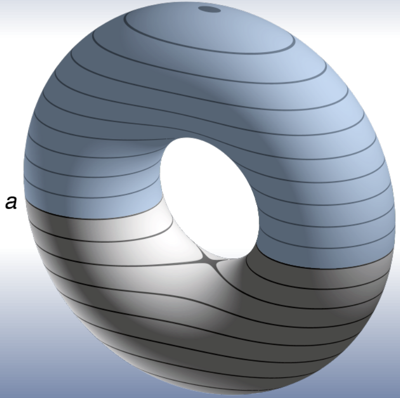


Define

$$M_a = \{x \in M^n : f(x) \leq a\}$$

for $a \in \mathbb{R}$

Theorems



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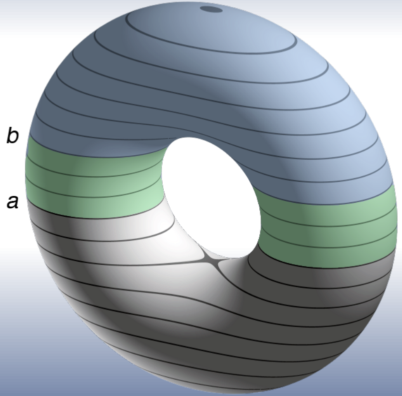

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Theorem (1)

Let $f^{-1}([a, b])$ has no critical points. Then

- *M_a and M_b are diffeomorphic*
- *$f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic*

Theorems

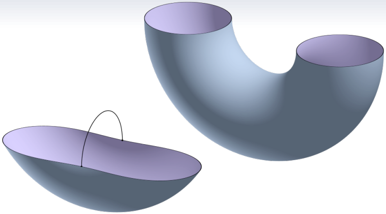


Figure: Gluing 1-cell in a place of critical point of index 1

Theorem (2)

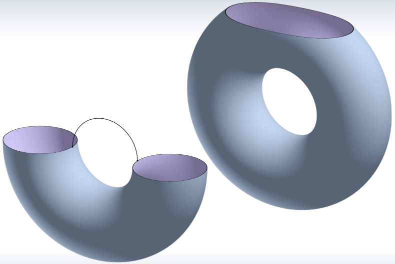
Let x is the only critical point of f in $f^{-1}([a,b])$ of index i, where

$$a = f(x) - \epsilon$$
$$b = f(x) + \epsilon$$

Then M_b is homotopy equivalent to $M_a \cup e^i$

e^i denotes i-cell (homomorphic to $int(D^i)$)

Theorems



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Theorems

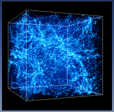
Note

Any Morse function can be perturbed to get different values for different critical points.
(to be injective on critical points set)

Theorems

Theorem (3)

If a Morse function f on M^n has c_i critical points of index i, then M^i is a homotopy equivalent of CW-complex containing exactly c_i cells of dimension i

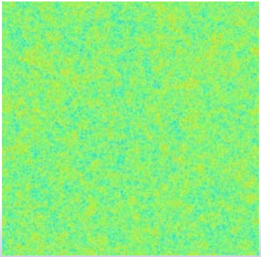


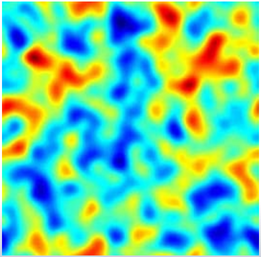
Cosmic Skeleton

Extracting from the cosmic web its filamentary pattern — draw in the observed structure a set of lines which repro- duces well the filamentary pattern guessed by eye [NCD06]

Gaussian smoothing

In this section we will use Gaussian random field $\rho(\mathbf{r})$ (left image) smoothed by convolution with Gaussian widow (right image).





[NCD06]

Curvature of filed

We suppose, that gradient disappears

$$\nabla\rho = \sum \frac{\partial\rho}{\partial r_i}\hat{\mathbf{r}}_{\mathbf{i}} = \mathbf{0}$$

only on discrete set of critical points, and for every point its Hessian

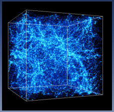
$$\mathbf{H}_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

is not degenerated. Taking Hessian eigenvalues $\lambda_1 \leq \lambda_2$ we have

$0 > \lambda_1 \geq \lambda_2$ for local maxima

$\lambda_1 > 0 > \lambda_2$ for saddle point

$\lambda_1 \geq \lambda_2 > 0$ for local minima



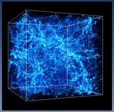
Cosmic Skeleton

void patches: regions in which all points conve to the same local minima, following gradient

$$\nabla\rho = \frac{\partial\rho}{\partial r_i}$$

the skeleton: borders of void patches

note: skeleton passes through all peaks and saddle points



Cosmic Skeleton

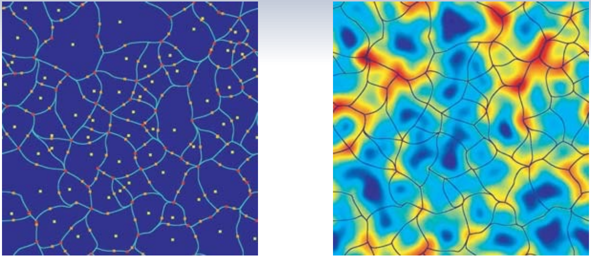
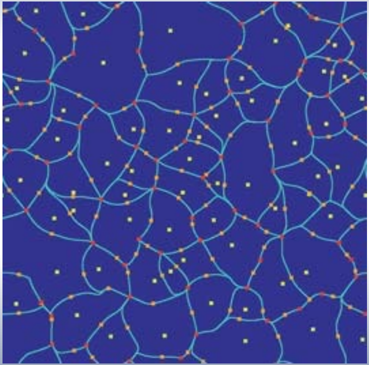


Figure: Skeleton of previous Gaussian field. Left panel: minima are in yellow, saddle points in orange and local maxima in brown. [NCD06]

Cosmic Skeleton

Skeleton has following properties

- Nodes of skeleton — where multiple skeleton lines can converge, are on local maxima
- Two local maxima cannot be directly connected together; there is always a saddle point in between
- Saddle points are not nodes of the skeleton. Only two field lines parallel to eigenvector corresponding to $\lambda_1 > 0$ (and end in local maxima) are the parts of an skeleton.

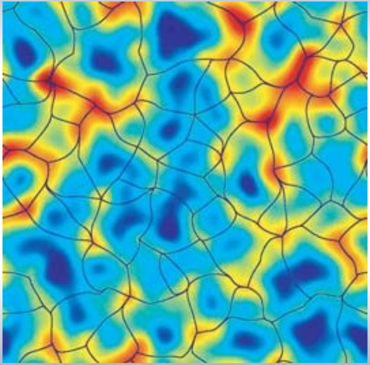


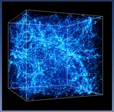
Drawing Skeleton

Skeleton can be seen as the ensemble of pairs of stable fields lines departing from saddle points and connecting them to local maxima. [NCD06]
On can draw skeleton, starting two lines form a saddle point, and following motion equation

$$\frac{d\mathbf{r}}{dt} = \nabla\rho$$

with initial velocity parallel to the eigenvector of the Hessian corresponding to positive λ_1 (major axis of the local curvature). Drawing stops on local maximum.

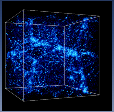




Limitations

Approach discussed in previous slides suffers from some limitations

- Not very practical because it is non-local and makes analytical predictions difficult. Solved by some local approximations in [NCD06].
- Introduces step of Gaussian smoothing loosing some information in favour of working on particular scale
- Identifies only single, specific scale structures, not taking into account multiscale nature of cosmic web. [ACPvdWS10]



the Watershed Transform [ACPvdWS10]

- Determination of **basins** which collects rainfall.
- Basins appears on smallest scale, and then joins into bigger structures in larger scales, when the water level reach saddle points.

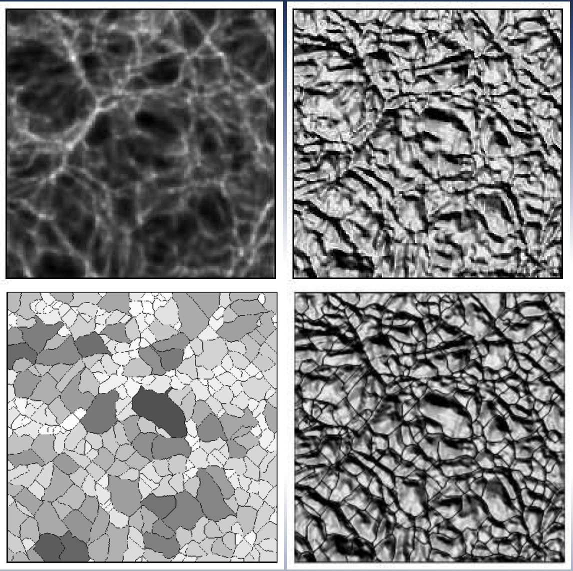
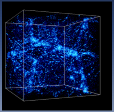


Figure: Density field from a N-body simulation. Same field as shadowed landscape. Watershed transform. Composition of former two. [ACPvdWS10]



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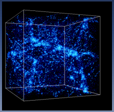
Basin is defined as as set of points x which are closer to particular minima y than to any other minima in the topological distance

$$T(x, y) = \int |\nabla f(\gamma(s))| \mathrm{d}s$$

along curve γ parallel to local gradient ∇f



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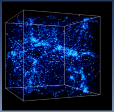
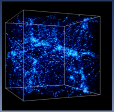


Discretess

Algorithm implementations faces discrete nature of input data and modelling process.
Discretness concerns

- spatial resolution — division of space into voxels (pixels in 3D)
- field f values — discrete number of values function can take

Both can aspects have influence, e.g. on assumptions taken in Morse theory analysis, which can cause problems in identifying critical points etc...



For Further Reading

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